UNIT-3
Vector spaces

Binary compositions:

Def-1: Internal Composition: let A be a set, then the mapping $f: A\times A \to A$ is called internal composition in it.

For Ex-1. $f: R\times R \to R$ defined as $f(x,y) = xy \;\forall (x,y)\in R$; x, y are Reals.

2. let A = set of all $n\times n$ matrices over reals.

if $f: A\times A \to A$ defined as
$f(P, Q) = P+Q \;\forall\; (P, Q) \in A\times A$; P, Q are $n\times n$ matrices over reals.

then f is an internal Composition in A.

Def-2 External Composition: let A and F be two non-empty sets. Then the mapping $f: A\times F \to A$ is called an external Composition on A by the elements of F.

For Ex- let A = set of all $n\times n$ matrices over reals.
F = set of all reals.

If $f: A\times F \to A$ is defined as $f(P, k) = kP \;\forall\; P\in A$ and $k\in F$

where kP means the multiplication of matrix P be the scalar k.

Then f is called external composition in A over F.

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vector space: let $(F, +, \cdot)$ be a given field and V be non empty set with two Compositions, '$+$' '$\cdot$' one is internal binary composition and other is external binary compositions. (one is addition, and other is multiplicative) Then the given set V is called a vector space or linear space over the field, iff the following axioms are satisfied

let R vectors with scal of

I. **Properties of Addition.**

A-1: Closure property: $\forall\ x, y \in V$, $x+y \in V$

A-2: Associative property: $\forall\ x, y, z \in V$, we have $(x+y)+z = x+(y+z)$

A-3: Existence of Additive identity:
$\exists$ an element $0 \in V$ s.t $x+0 = x = 0+x\ \forall\ x \in V$

A-4: Existence of Additive inverse:
for each $x \in V$ $\exists$ an element $-x \in V$ s.t

$$x + (-x) = 0 = (-x) + x.$$

where $-x$ is called additive inverse of x.

A-5: commutative property: $\forall\ x, y \in V$ we have $x+y = y+x$

II **Properties of scalar multiplication**

M-1: $\forall\ \alpha \in F,\ x \in V$ we have $\alpha x \in V$

M-2: $\forall,\ \alpha, \beta \in F,\ x \in V$ we have $(\alpha+\beta)x = \alpha x + \beta x$

M-3: $\forall\ \alpha \in F,\ x, y \in V$ we have $\alpha(x+y) = \alpha x + \alpha y$

M-4: $\forall\ \alpha, \beta \in F,\ x \in V$ we have $(\alpha\beta)x = \alpha(\beta x)$

M-5: $\forall\ x \in V$ we have $1.x = x = x.1$, where 1 is the unity element of F.

let R be the field of reals and V be the set of vectors in a plane. Show that $V(R)$ is a vector space with vector addition as internal binary composition and scalar multiplication of the elements of R with those of V as external binary composition.

Sol: Given $V = \{(x, y) / x, y \in R\}$

Here we define addition of vectors in V as

$$(x, y) + (t, z) = (x+t, y+z) \quad \forall\ x, y, t, z \in R$$

and scalar multiplication of $\alpha \in R$ and $(x, y) \in V$ as

$$\alpha(x, y) = (\alpha x, \alpha y)$$

Properties under Addition:

A-1: Closure: let $(x_1, y_1), (x_2, y_2) \in V \Rightarrow x_1, y_1, x_2, y_2 \in R$

$\Rightarrow x_1 + x_2, y_1 + y_2 \in R$

$\therefore (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in V$

$\Rightarrow$ V is closed under Addition.

A-2 Associative: let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$

Now $[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = [(x_1 + x_2), (y_1 + y_2)] + (x_3, y_3)$

$= (x_1 + x_2 + x_3), (y_1 + y_2 + y_3)$

$= [x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)]$

$= (x_1, y_1) + [(x_2 + x_3), (y_2 + y_3)]$

$= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)].$

Addition is associative in V.

(4)

A:3 Existence of additive identity: for all $(x_1, y_1) \in V$: let There exist $(0,0) \in V$ s.t

$$(x_1, y_1) + (0,0) = (x_1+0, y_1+0) = (x_1, y_1)$$

and $(0,0) + (x_1, y_1) = (0+x_1, 0+y_1) = (x_1, y_1)$.

$\Rightarrow$ $(0,0)$ is additive identity in V.

A:4 Existence of additive inverse: Now let $(x, y) \in V$

$\Rightarrow (-x, -y) \in V$ $\left\{ \because x, y \in R \Rightarrow -x, -y \in R \right\}$

Now $(x, y) + (-x, -y) = (x-x, y-y) = (0,0)$

and $(-x, -y) + (x, y) = (-x+x, -y+y) = (0,0)$

$\therefore (x, y) + (-x, -y) = (0,0) = (-x, -y) + (x, y)$

$\Rightarrow (-x, -y)$ is additive inverse of (x, y) for each $(x, y) \in V$.

A:5 Commutative: let $(x_1, y_1), (x_2, y_2) \in V$

Now $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2), (y_1 + y_2)$

$= (x_2 + x_1), (y_2 + y_1)$ --- $\left\{ \because \text{Addition is Commutative in Real no.} \right\}$

$= (x_2, y_2) + (x_1, y_1)$.

$\therefore$ Addition is commutative in V.

Properties under scalar multiplication

M-1: let $\alpha \in R$, $(x, y) \in V$; $x, y \in R$

Then $\alpha(x, y) = (\alpha x, \alpha y) \in V$ $\left\{ \because \alpha \in R, x, y \in R \text{ then } \alpha x, \alpha y \in R \right\}$

M-2: let $\alpha \in R$ and $(x_1, y_1), (x_2, y_2) \in V$.

Now $\alpha[(x_1, y_1) + (x_2, y_2)] = \alpha[x_1 + x_2, y_1 + y_2] = [\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2]$

$= (\alpha x_1, \alpha y_1)\ (\alpha x_2, \alpha y_2)$

$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$.

M_3: let $\alpha, \beta \in R$ and $(x_1, y_1) \in V$

Now $(\alpha+\beta)(x_1, y_1) = ((\alpha+\beta)x_1, (\alpha+\beta)y_1)$

$= (\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_1)$

$= (\alpha x_1, \alpha y_1) + (\beta x_1, \beta y_1)$

$= \alpha(x_1, y_1) + \beta(x_1, y_1)$

M-4: let $\alpha, \beta \in R$ and $(x_1, y_1) \in V$

Now $(\alpha\beta)(x, y) = (\alpha\beta x, \alpha\beta y)$

$= [\alpha(\beta x), \alpha(\beta y))$

$= \alpha(\beta x, \beta y)$

$= \alpha(\beta(x, y))$

M-5: let $1 \in R$ and $(x_1, y_1) \in V$

Now $1.(x_1, y_1) = (1.x_1, 1.y_1) = (x_1, y_1)$

Hence V is a vector space over R.

(2). let V be set of all real valued continous (differentiable or integrable) functions defined in closed interval $[a, b]$, then show that V is a vector space R with addition and scalar multiplication defined as

$(f+g)(x) = f(x) + g(x) \quad \forall f, g \in V$

and $(\alpha f)x = \alpha f(x) \quad \forall \alpha \in R, f \in V$.

Sol: Given $V = \{f \mid f$ is real valued continous function defined on $[a, b]\}$

Properties under addition

A-1: Closure. let $f, g \in V$

$\Rightarrow f, g$ are real valued continous function on $[a, b]$

and $(f+g)x = f(x) + g(x) \quad \forall \; x \in [a, b]$

Since $f(x)$ and $g(x)$ are real valued continous functions on [

and sum of two real valued continous functions is a real valu

continous function so $f(x) + g(x)$ is a real valued continous

function on $[a, b]$

$\Rightarrow (f+g)x$ is a real valued continous function on $[a, b]$

$\Rightarrow f+g \in V$ for all $f, g \in V$

Thus V is closed under addition.

A-2: Associative: let $f, g, h \in V$ and $x \in [a, b]$

Now $[(f+g)+h](x) = (f+g)x + h(x)$

$= f(x) + g(x) + h(x)$

$= f(x) + [g(x) + h(x)]$

$= f(x) + (g+h)x$

$= [f + (g+h)]x \quad \forall \; x \in [a, b]$

$\therefore (f+g)+h = f + (g+h)$

$\Rightarrow$ addition is associative in V.

A-3: Existence of additive identity: Defined a function 0 on $[a, b]$

s.t $0(x) = 0 \quad \forall \; x \in [a, b]$

Thus 0 is a real valued continous function on $[a, b]$

$\Rightarrow 0 \in V$

Now for all $f \in V$, $x \in [a, b]$

$(0 + f)(x) = 0(x) + f(x) = 0 + f(x) = f(x)$

$\Rightarrow (0+f)x = f(x) \quad \forall \; x \in [a,b]$

$\Rightarrow \quad 0+f = f$

and $(f+0)x = f(x) + 0(x) = f(x) + 0 = f(x)$

$\Rightarrow (f+0)x = f(x) \quad \forall \; x \in [a,b]$

$\Rightarrow \quad f+0 = f$

Thus $0+f = f = f+0$

$\Rightarrow$ 0 is the addative identity.

A-4: Existence of addative inverse: For each $f \in V$, we defined $-f \in V$ as $(-f)(x) = -f(x) \quad \forall \; x \in [a,b]$

$\Rightarrow$ $-f$ is real valued continuous function in V

$\Rightarrow \quad -f \in V$

Now $[f+(-f)]x = f(x) + (-f)x = f(x) - f(x) = 0 = 0(x)$

$\Rightarrow \quad f+(-f) = 0 \quad \forall \; f \in V$

||ly $[(-f)+f]x = 0x \quad \Rightarrow \quad (-f)+f = 0 \quad \forall \; f \in V$

$\therefore \quad f+(-f) = 0 = (-f)+f \quad \forall \; f \in V$

$\Rightarrow$ $-f$ is the addative inverse of f.

A-5: Commutative let $f, g \in V$

Now $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)x$

$\therefore \quad (f+g)x = (g+f)x \quad \forall \; x \in [a,b]$

$\Rightarrow \quad f+g = g+f \quad \forall f, g \in V$

$\Rightarrow$ addition is commutative in V.

$\bar{\text{II}}$ Properties under scalar multiplication:

M-1: Let $\alpha \in R$ and $f \in V$

Now $(\alpha f)x = \alpha f(x) \quad \forall x \in [a,b]$

$\Rightarrow \alpha f \in V$ {$\because$ a scalar multiple of real valued continuous function is a real valued continuous function}

M:2: Let $\alpha \in R$ and $f, g \in V$

Now $[\alpha(f+g)]x = \alpha[(f+g)x]$

$= \alpha[f(x) + g(x)]$

$= \alpha f(x) + \alpha g(x)$

$= (\alpha f)x + (\alpha g)x$

$= (\alpha f + \alpha g)x \quad \forall x \in [a,b]$

$\Rightarrow \alpha(f+g) = \alpha f + \alpha g$

M-3: Let $\alpha, \beta \in R$ and $f \in V$

Now $[(\alpha+\beta)f]x = (\alpha+\beta)f(x) \quad \forall x \in [a,b]$

$= \alpha f(x) + \beta f(x)$

$= (\alpha f)x + (\beta f)x = (\alpha f + \beta f)x$

$\Rightarrow (\alpha+\beta)f = \alpha f + \beta f.$

M-4: Let $\alpha, \beta \in R$ and $f \in V$

Now $[(\alpha\beta)f](x) = (\alpha\beta)f(x)$

$= \alpha(\beta f(x))$

$= \alpha[(\beta f)(x)]$

$= [\alpha(\beta f)] \quad \forall x \in \forall x \in [a,b]$

$\Rightarrow (\alpha\beta)f = \alpha(\beta f)$

M-5: Let $1 \in R$ and $f \in V$

Now $(1 \cdot f)(x) = 1 \cdot f(x) = f(x) \quad \forall x \in [a,b]$

$\Rightarrow 1 \cdot f = f \quad \forall f \in V$

Hence V is a vector space over R. Proved.

3. Let V set of all real valued continous functions defined on $[0,1]$ such that $f(\frac{2}{3}) = 2$. Show that V is not a vector space over R (reals) under addition and scalar multiplication defined as

$$(f+g)x = f(x) + g(x) \quad \forall\ f, g \in V$$
$$(\alpha f)x = \alpha f(x) \quad \forall\ \alpha \in R,\ f \in V$$

Sol:- Let $f, g \in V$

$\Rightarrow$ f and g are real valued continous functions defined on $[0,1]$ such that $f(\frac{2}{3}) = 2$ and $g(\frac{2}{3}) = 2$

Now $(f+g)(\frac{2}{3}) = f(\frac{2}{3}) + g(\frac{2}{3})$

$$= 2+2$$
$$= 4$$

$\Rightarrow$ $f+g \notin V$ where $f, g \in V$

For ex- let $f(x) = 3x$ $\quad x \in [0,1]$

$\Rightarrow$ f is real valued continous function defined on $[0,1]$

and $f(\frac{2}{3}) = 3(\frac{2}{3}) = 2$ so $f \in V$

let $g(x) = 2$ $\quad x \in [0,1]$

$\Rightarrow$ g is real valued continous function defined on $[0,1]$

and $g(\frac{2}{3}) = 2$

Now $f+g$ is a real valued continous function

But $(f+g)(\frac{2}{3}) = f(\frac{2}{3}) + g(\frac{2}{3}) = 2+2 = 4$

$\Rightarrow$ $f+g \in V$ where as $f, g \in V$

$\Rightarrow$ closure property does not hold.

Hence V is not vector space over R.

Some other Problems:

(4) If $P(x)$ is the set of all polynomials in one indeterminate x over a field F. then show that $P(x)$ is a vector space over F with addition defined as addition of polynomials and scalar multiplication defined as product of polynomial by an element of F.

Sol: Hint Given $P(x) = \{ f(x) / f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots \alpha_n x^n + \dots$

$= \{ f(x) / f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ for α_k's $\in F \}$

we define addition and multiplication as

if $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, $g(x) = \sum_{k=0}^{\infty} \beta_k x^k \in P(x)$

Then $f(x) + g(x) = \sum_{k=0}^{\infty} (\alpha_k + \beta_k) x^k$

and $a f(x) = \sum_{k=0}^{\infty} (a\alpha_k) x^k$ for $a \in F$.

(5) Prove that any field forms a vector space over itself.

(6) Show that the set of all matrices of form $\begin{bmatrix} a & b \\ -b & \end{bmatrix}$ where $a, b \in C$ is a vector space over C under matrix addition and scalar multiplication.

Linear Dependence and Linear Independence.

Linear Dependence (L.D):- If V be a vector space over Field F, then the vectors $v_1, v_2, \ldots v_n \in V$ is called linearly dependent over F if $\exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_n \in F$ not all of them zero (ie at least one of α_i's is non zero) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0$$

Linear Independent (L.I) If V is a vector space of a field F, then the vectors $v_1, v_2, \ldots v_n \in V$ are called Linearly independent over F if $\exists$ scalars $\alpha_1, \alpha_2 \ldots \alpha_n \in F$ all of them zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots \alpha_n v_n = 0.$$

Linear Combination: let V be a vector space over F,

If $v_1, v_2, \ldots v_n \in V$

then any element v written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$

(where α_i's $\in F$, $1 \le i \le n$)

is called Linear combination of all vectors. $v_1, v_2 \ldots v_n$ over F.

Q:-1: If v is a Linear combination of $v_1, v_2 \ldots v_n$ then Show that $v_1, v_2 \ldots v_n, v$ are L.D vectors.

Sol: Given v is a linear combination of $v_1, v_2 \ldots v_n$

$\Rightarrow \exists$ scalar $\alpha_1, \alpha_2 \ldots \alpha_n$ s.t

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n - v = 0$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha v = 0 \qquad \text{where } \alpha = -1 \neq 0$$

$\therefore \exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_n, \alpha$ not all zero s.t

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n + \alpha v = 0 \Rightarrow v_1, v_2 \ldots v_n, v$ are L.D

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Q:-2: If u, v, w are L.I vectors in a vector space then show that

(a) the vectors $u+v$, $v+w$, $w+u$ are L.I.

(b) The vectors $u+v$, $u-v$, $u-2v+w$ are L.I.

Sol: let $\alpha_1, \alpha_2, \alpha_3$ be scalars in F such that

$$\alpha_1(u+v)+\alpha_2(v+w)+\alpha_3(w+u)=0$$

$$\Rightarrow (\alpha_1+\alpha_3)u+(\alpha_1+\alpha_2)v+(\alpha_2+\alpha_3)w=0$$

{$\because u, v, w$ are L.I}

$\Rightarrow \alpha_1+\alpha_3=0$ ——(1)

$\alpha_1+\alpha_2=0$ ——(2)

$\alpha_2+\alpha_3=0$ ——(3)

Adding (1) + (2) + (3), $2(\alpha_1+\alpha_2+\alpha_3)=0$

ie $\alpha_1+\alpha_2+\alpha_3=0$ ——(4)

From (1) and (4), $\alpha_2=0$ ——(5)

From (2) and (5) $\alpha_1=0$ ——(6)

and from (1) and (6) $\alpha_3=0.$ ——(7)

Hence $\alpha_1=\alpha_2=\alpha_3=0$,

$\Rightarrow u+v$, $v+w$, $w+u$ are L.I.

(b) let $\alpha_1, \alpha_2, \alpha_3 \in F$ s.t

$$\alpha_1(u+v)+\alpha_2(u-v)+\alpha_3(u-2v+w)=0$$

$$(\alpha_1+\alpha_2+\alpha_3)u+(\alpha_1-\alpha_2-2\alpha_3)v+\alpha_3 w=0$$

Since u, v, w are L.I.

$\therefore \alpha_1+\alpha_2+\alpha_3=0$

$\alpha_1-\alpha_2-2\alpha_3=0$

$\alpha_3=0$

find the value of $\alpha_1, \alpha_2, \alpha_3$ we get $\alpha_1=\alpha_2=\alpha_3=0$

$\therefore$ Given vectors are L.I.

Prove that system of vectors
$u = (1, 2, -3)$; $v = (1, -3, 2)$ and $w = (2, -1, 5)$
of $V_3(R)$ is L.I.

Sol: let $\alpha_1, \alpha_2, \alpha_3 \in R$ s.t

$$\alpha_1 u + \alpha_2 v + \alpha_3 w = 0$$

$$\Rightarrow \alpha_1(1, 2, -3) + \alpha_2(1, -3, 2) + \alpha_3(2, -1, 5) = (0, 0, 0)$$

$$\Rightarrow (\alpha_1 + \alpha_2 + 2\alpha_3,\ 2\alpha_1 - 3\alpha_2 - \alpha_3,\ -3\alpha_1 + 2\alpha_2 + 5\alpha_3) = (0, 0, 0)$$

$$\therefore \quad \alpha_1 + \alpha_2 + 2\alpha_3 = 0$$
$$2\alpha_1 - 3\alpha_2 - \alpha_3 = 0$$
$$-3\alpha_1 + 2\alpha_2 + 5\alpha_3 = 0$$

In matrix form such as $AX = 0$

~~where A~~ ie $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ 3 & 2 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ 3 & 2 & 5 \end{vmatrix} = 1(-15+2) - 1(10-3) + 2(4-9)$

$$= -30 \neq 0$$

$\therefore$ Equation have a trivial sol.

ie $\alpha_1 = \alpha_2 = \alpha_3 = 0$

Hence the vectors u, v, w are L.I.

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(14).

(4)

let $x = (2, -1, 0)$; $y = (1, 2, 1)$ and $z = (0, 2, -1)$, or $(3) \times 2$ $-\alpha$

Show that x, y, z are linearly independent.

Express $(3, 2, 1)$ as a linear combination of x, y, z.

Sol: let $\alpha_1, \alpha_2, \alpha_3 \in F$

Such that $\alpha_1 x + \alpha_2 y + \alpha_3 z = 0$

ie $\alpha_1 (2, -1, 0) + \alpha_2 (1, 2, 1) + \alpha_3 (0, 2, -1) = (0, 0, 0)$

$\Rightarrow (2\alpha_1 + \alpha_2, -\alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 - \alpha_3) = (0, 0, 0)$

ie
$$2\alpha_1 + \alpha_2 = 0 \quad \text{——} (1)$$
$$-2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 \quad \text{——} (2)$$
$$\alpha_2 - \alpha_3 = 0 \quad \text{——} (3)$$

These Equation can be written as $AX = O$

where $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ $X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$|A| = -9 \neq 0$

$\therefore$ Equation (1), (2), (3) have a trivial solution

ie $\alpha_1 = \alpha_2 = \alpha_3 = 0$

Hence the given vectors are L.I.

IInd part: let $v = \alpha_1 x + \alpha_2 y + \alpha_3 z$ for some scalar $\alpha_1, \alpha_2, \alpha_3 \in F$

$\Rightarrow (3, 2, 1) = \alpha_1 (2, -1, 0) + \alpha_2 (1, 2, 1) + \alpha_3 (0, 2, -1)$

$= (2\alpha_1 + \alpha_2, -\alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 - \alpha_3)$

By Equality of vectors

$$2\alpha_1 + \alpha_2 = 3 \quad \text{——} (1)$$
$$-\alpha_1 + 2\alpha_2 + 2\alpha_3 = 2 \quad \text{——} (2)$$
$$\alpha_2 - \alpha_3 = 1 \quad \text{——} (3)$$

From (3)×2 + (2) gives

$-\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_2 - 2\alpha_3 = 2+2$

$\Rightarrow -\alpha_1 + 4\alpha_2 = 4$ ——— (4)

from (4)×2 + (1), we get

$9\alpha_2 = 11 \Rightarrow \alpha_2 = \frac{11}{9}$

using in (1), we get $\alpha_1 = \frac{8}{9}$

using the value of α_2 in (3), we get $\alpha_3 = \frac{2}{9}$.

Hence $\boxed{v = \frac{8}{9}x + \frac{11}{9}y + \frac{2}{9}z}$ Ans.

(5). Prove that the following system of vectors of $V_3(R)$ are L.D.

$x = (1,3,2)$; $y = (1,-7,-8)$; $z = (2,1,-1)$

Sol: let $\alpha_1, \alpha_2, \alpha_3 \in F$ s.t

$\alpha_1 x + \alpha_2 y = \alpha_3 z = 0$

ie $\alpha_1(1,3,2) + \alpha_2(1,-7,-8) + \alpha_3(2,1,-1) = (0,0,0)$

ie $\alpha_1 + \alpha_2 + 2\alpha_3 = 0$

$3\alpha_1 - 7\alpha_2 - \alpha_3 = 0$

$2\alpha_1 - 8\alpha_2 - \alpha_3 = 0$

These equations can be matrix form as $AX = O$

where $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & -1 \\ 2 & -8 & -1 \end{bmatrix}$ $X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & -7 & -1 \\ 2 & -8 & -1 \end{vmatrix} = 0$

$\Rightarrow$ The equations have a non-trivial solution.

$\therefore \alpha_1, \alpha_2, \alpha_3$ are not all zero. Hence x, y, z, are L.D.

Some Problems: (Based upon L.D and L.I)

1. Prove that the following system of vectors in $V_3(R)$ are L.D.

(a) $x = (3,0,-3)$; $y = (-1,1,2)$, $z = (4,2,-2)$, $w = (2,1,1)$

(2) Prove that the following system of vectors of $V_3(R)$ are L.I.

(a) $x = (1,5,2)$; $y (0,0,1)$; $z = (1,1,0)$

(b) $x = (1,-1,2,0)$; $y = (1,1,2,0)$; $z = (3,0,0,1)$ and $w = (2,1,-1,0)$.

Linear span

If S is a non empty subset of vector space $V(F)$, then the set of all linear combinations of any finite number of elements of S is called the linear span of S. The linear span of S is denoted by $L(S)$

So that $L(S) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \mid v_i \in S \text{ and } \alpha_i \in F,\ 1 \le i \le n \right\}$

Note:- If $S = \phi$ then $L(S) = \{0\}$

Basis:

Def: Let $V(F)$ be a vector space. A subset B of V is called a basis of V iff

(i) B is Linearly Independent set.

(ii) $L(B) = V$ ie B generates (spans) V.

or in other words every element in V is a linear combination of the element of B.

Note: 1. A set of vectors having zero vector is always L.D. set, so it cannot basis of vector space. Thus a zero vector cannot be an element of basis of a vector space.

2. Since $L(\phi) = \{0\}$ and ϕ is L.I

$\therefore$ ϕ is a basis of $\{0\}$

3. $\{0\}$ is not a basis of $\{0\}$

Q:-1 Show that the set $B = \{e_1, e_2, \dots e_n\}$ where $e_i = (0, 0, \dots 1, 0, \dots 0)$ occurs at ith place is a basis of $V_n(R)$.

Sol: We know that

$$V_n(R) = \{ v / v = (\alpha_1, \alpha_2 \dots \alpha_n) ; \alpha_i \in F \text{ where } 1 \le i \le n \}$$

(I) To Prove B is L.I. Set

Let $\alpha_1 e_1 + \alpha_2 e_2 + \dots \alpha_n e_n = 0$ for α_i's $\in F$

$\Rightarrow \alpha_1(1, 0, 0, \dots) + \alpha_2(0, 1, 0, \dots 0) + \dots \alpha_n(0, 0, \dots 1) = (0, 0, \dots 0)$

$\Rightarrow (\alpha_1, \alpha_2, \dots \alpha_n) = (0, 0, 0, \dots 0)$

ie $\alpha_1 = \alpha_2 = \dots \alpha_n = 0$ ie $e_1, e_2, \dots e_n$ are L.I.

II. To Prove $L(B) = V_n(R)$

let $v \in V_n(R) \Rightarrow v = (\alpha_1, \alpha_2 \dots \alpha_n)$ for α_i's $\in F$ $1 \le i \le n$

$\Rightarrow v = (\alpha_1, 0, \dots 0) + (0, \alpha_2, \dots 0) + \dots + (0, 0, \dots \alpha_n)$

$\Rightarrow v = \alpha_1 (1, 0, \dots 0) + \alpha_2 (0, 1, 0, \dots 0) + \dots \alpha_n (0, 0, \dots 1)$

$= \alpha_1 e_1 + \alpha_2 e_2 + \dots \alpha_n e_n$

$\Rightarrow v$ is linear combination of $e_1, e_2, \dots e_n$ the elements of B.

$\Rightarrow v \in L(B)$

$\therefore V_n(R) \subset L(B)$ —— (1)

Also $L(B) \subset V_n(R)$ —— (2)

from (1) and (2) $L(B) = V_n(R)$

Hence B is basis of $V_n(R)$.

Q:-2:- Give Example of two different basis of $V_2(R)$.

Sol: we know $V_2(R) = \{(\alpha, \beta) / \alpha, \beta \in R\}$

Consider the sets $B_1 = \{(1,0), (0,1)\} = \{e_1, e_2\}$

and $B_2 = \{(2,3), (1,2)\} = \{v_1, v_2\}$

we show that the sets B_1 and B_2 both form basis for $V_2(R)$

(i) To show that B_1 is L.I.

let $\alpha_1 e_1 + \alpha_2 e_2 = 0$ for $\alpha_1, \alpha_2 \in R$

$\Rightarrow \alpha_1 (1, 0) + \alpha_2 (0, 1) = (0, 0)$

$\Rightarrow (\alpha_1, \alpha_2) = (0, 0)$

ie $\alpha_1 = 0$, $\alpha_2 = 0$. $\Rightarrow e_1, e_2$ are L.I vectors.

ii To show that $L(B_1) = V_2 R$

let $v \in V_2(R)$ Then $v = (\alpha, \beta)$ for $\alpha, \beta \in R$

$= (\alpha, 0) + (0, \beta)$

$= \alpha(1, 0) + \beta(0, 1)$

$= \alpha e_1 + \beta e_2$

$\Rightarrow v$ is a linear combination of e_1 and e_2, the elements of set B_1

$\therefore \; v \in L(B_1) \Rightarrow V_2(R) \subset L(B_1)$ — (1)

Also $L(B_1) \subset V_2(R)$ — (2)

From (1) and (2) $L(B_1) = V_2(R)$

Hence B_1 is Basis for $V_2(R)$.

Now To show that B_2 is L.I

let $\alpha_1 v_1 + \alpha_2 v_2 = 0$ for $\alpha_1, \alpha_2 \in R$

$\Rightarrow \alpha_1(2,3) + \alpha_2(1,2) = (0,0)$

$\Rightarrow \quad 2\alpha_1 + \alpha_2 = 0$ — (1)

and $\quad 3\alpha_1 + 2\alpha_2 = 0$ — (2)

Solving, $\alpha_1 = 0, \quad \alpha_2 = 0$

$\therefore$ B_2 is L.I set.

To Show $L(B_2) = V_2(R)$

let $v = (\alpha, \beta)$ for $\alpha, \beta \in R$

we shall express v as a Linear combination of v_1 and v_2

Suppose $v = \alpha_1 v_1 + \alpha_2 v_2$ for some $\alpha_1, \alpha_2 \in R$

Now $(\alpha, \beta) = \alpha_1(2,3) + \alpha_2(1,2)$

$= (2\alpha_1 + \alpha_2, \; 3\alpha_1 + 2\alpha_2)$

$\therefore \quad 2\alpha_1 + \alpha_2 = \alpha$

$3\alpha_1 + 2\alpha_2 = \beta$

Solving, $\alpha_1 = 2\alpha - \beta$

and $\alpha_2 = 2\beta - 3\alpha$

$\therefore v = (\alpha, \beta) = (2\alpha - \beta)(2,3) + (2\beta - 3\alpha)(1,2)$

$= (2\alpha - \beta)v_1 + (2\beta - 3\alpha)v_2$

$\Rightarrow v$ is L.C of v_1 and v_2 the element of B_2

$\therefore v \in L(B_2)$ ie $V_2(R) \subset L(B_2)$ Also $L(B_2) \subset V_2(R)$

ie $L(B_2) = V_2(R)$ Hence B_2 is a basis of $V_2(R)$.

Q:-3: Give an examples of two different basis of $V_3(R)$ ↑

Sol:- (Hint):- $V_3(R) = \{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in R\}$

Consider the sets $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\} = \{e_1, e_2, e_3\}$

and $B_2 = \{(0,1,0), (0,0,1), (2,3,4)\} = \{v_1, v_2, v_3\}$

We show the sets B_1 and B_2 both form Basis for $V_3(R)$.

Q:-4: Show that $B = \{(1,1,1), (1,-1,1), (0,1,1)\}$ is a basis of R^3.

(Note: Theorem: A subset W of V having n elements is a basis iff W is L.I iff $L(W) = V$)

∴ This result is used in above Question.

Sol: As $\dim R^3 = 3$ Thus to show B is a basis of R^3, It is sufficient to check B is L.I set.

let $\alpha_1(1,1,1) + \alpha_2(1,-1,1) + \alpha_3(0,1,1) = 0$ for $\alpha_1, \alpha_2, \alpha_3 \in R$

$\Rightarrow (\alpha_1 + \alpha_2 + 0\alpha_3, \alpha_1 - \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3) = (0, 0, 0)$

$$\therefore \quad \alpha_1 + \alpha_2 = 0$$
$$\alpha_1 - \alpha_2 + \alpha_3 = 0$$
$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

These equation can be written (in matrix form) as

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = 0$ where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Now $\det A = |A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2 \neq 0$

$\therefore$ System of linear equation has trivial sol.

ie $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ B is L.I set

Hence B is a basis of R^3.

Q:-5 Examine whether the set of vectors in $V_3(R)$ forms a basis or not

$(1, \frac{2}{5}, -1), (0, 1, 2), (\frac{3}{4}, -1, 1)$

Sol. As $\dim V_3(R) = 3$ Thus to show B_3 is basis or not basis of $V_3(R)$. ie we check out B_3 is L.I or L.D. set.

let $\alpha_1(1, \frac{2}{5}, -1) + \alpha_2(0, 1, 2) + \alpha_3(\frac{3}{4}, -1, 1) = (0, 0, 0)$

when $\alpha_1, \alpha_2, \alpha_3 \in R$

$\Rightarrow (\alpha_1 + \frac{3}{4}\alpha_3, \frac{2}{5}\alpha_1 + \alpha_2 - \alpha_3, -\alpha_1 + 2\alpha_2 + \alpha_3) = (0, 0, 0)$

$\therefore \alpha_1 + \frac{3}{4}\alpha_3 = 0$ —— (1)

$\frac{2}{5}\alpha_1 + \alpha_2 - \alpha_3 = 0$ —— (2)

$-\alpha_1 + 2\alpha_2 + \alpha_3 = 0$ —— (3)

This equation can be put in matrix form as

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} \\ \frac{2}{5} & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ie $AX = 0$

where $A = \begin{bmatrix} 1 & 0 & 3/4 \\ 2/5 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$ $|A| = \frac{87}{20} \neq 0$

$\therefore$ Eqs (1), (2), (3) have only trivial sol.

ie $\alpha_1 = \alpha_2 = \alpha_3 = 0$, ie B_3 is basis of $V_3(R)$.

Prob: 1. Examine the set of vectors (1, 0, 0), (0, 1, 0), (1, 1, 0) (1,
in $V_3(R)$ forms a basis or not.

Sol: (Hint: Check out the these sets. wether L.I or L.D.)
and these sets of vectors are L.D.
$\therefore$ Given set of vectors are not Basis.

2. Show that the vectors (1, 1, 1) (1, 0, 1) (1, −1, −1) of R^3 form a basis of $R^3(R)$. Also find the co-ordinate vector of (−3, 5, 7) relative to this basis.

Sol. (Hint): Check given vectors are L.I. or L.D.
These sets of vectors are L.I
$\therefore$ Given vectors form a basis of $R^3(R)$

IInd part: Let $\alpha, \beta, \gamma \in R$ s.t

$$(-3, 5, 7) = \alpha(1, 1, 1) + \beta(1, 0, 1) + \gamma(1, -1, -1)$$

find the value of α, β, γ

we get. $\alpha = 0, \quad \beta = 2, \quad \gamma = -5$

$$\therefore (-3, 5, 7) = 0(1, 1, 0) + 2(1, 0, 1) + (-5)(1, -1, -1)$$

$\Rightarrow$ The required co-ordinate vector is $(0, 2, -5)$.

(1,1,0) (1,

Dimension

nite Dimensional: A vector space $V(F)$ is called finite dimensional or finitely generated iff there exits a finite subset S of V such that $L(S)$ i.e. linear span of S is equal to V.

* If there exit no finite subset which generates V, then V is called an infinite dimensional vector space

Dimension of vector space: The dimension of a finitely generated vector $V(F)$ is defined as the number of elements in a basis of $V(F)$ and is denoted by $\dim V$.

i.e. If any basis of V contains n elements we say $\dim V = n$ and Thus V is n-dimensional vector space.

for example If $V = R^n$ then $\dim V = n$

Note: (i) $\dim\{0\} = 0$, as basis of zero space is empty set which contains no element.

(ii) If vector space $V(F)$ is not a finitely generated vector space, then it is called to be infinite dimensional vector space and $\dim V = \infty$.

\# Under what conditions on the scalar b, do the vectors $(1,1,1)$ and $(1, b, b^2)$ form a basis of $V_3(C)$?

Sol: As $\dim V_3(C) = 3$

$\therefore$ the basis of $V_3(C)$ will contain exactly three vectors.

Hence the set $\{(1,1,1), (1, b, b^2)\}$ cannot be the basis of $V_3(C)$

$\therefore$ For no value of b, the given vectors form a basis of $V_3(C)$

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Q:- ~~Find~~ Find a basis and solution space S of the following System of linear equations

$$x + y - 2z + 2s - t = 0$$
$$x + 2y - z + 3s - 2t = 0$$
$$2x + 4y - 7z + s + t = 0$$

Sol: The given system of equations can be written as $AX = B$ —(1)

where $A = \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $A = \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 2 & -3 & -3 & 3 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 5 & 0 & 0 & 0 \end{bmatrix} \quad \left(R_3 \to \tfrac{1}{5}R_3\right) \sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(i) $R_1 \to R_1 - R_2$
(ii) $R_2 \to R_2 - R_3$

$\therefore$ Equation (1) becomes

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - 3z + s = 0$ (2), $z + s - t = 0$ (3), $\boxed{y = 0}$

let $z = K$, $t = s$, $\Rightarrow s = s - K$. Putting in (2), we get

$x - 3K + s - K \Rightarrow x = 4K - s = -s + 4K$

$$\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} K$$

Here dim of solution set ie dim $S = 2$ and Basis of sol. set is $\{(-1,0,0,1,1), (0,0,1,-1,0)\}$

$\therefore$ dimension $= 2$

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